Math 3235 Probability Theory

2/7/2023

Section 3 Multivariate discrete distributions.

X is a discrete r.v.

$X: \Omega \to \mathbb{R}$

such that

$Im(X)$ is a discrete set.

$A_x = \{\omega \in \Omega \mid X(\omega) = x\} \in \mathcal{F}$

for every x.

You have more than 1 r.v.

For example: rolling two dice.

$\Omega = \{(i,j) \mid i,j = 1 \dots 6\}$

$X_1(i,j) = i$ The outcome of the first die

$X_2(i,j) = j$ the second

Define the joint p.m.f. of X_1 and X_2 as

$$p_{X_1, X_2}(x_1, x_2) = \mathbb{P}(X_1 = x_1 \text{ and } X_2 = x_2)$$

Two dice:

$$p(x_1, x_2) = \frac{1}{36} \quad \text{for}$$

$$1 \le x_1, x_2 \le 6$$

But I can also consider

$Y_1 = X_1 + X_2$ $\qquad Y_2 = X_1 - X_2$

Joint p.m.f. of Y_1 and Y_2?

$$\mathrm{Im}(Y_1) = \{2 \dots 12\}$$

$$\mathrm{Im}(Y_2) = \{-5, 5\}$$

$y_1 \in Im(Y_1)$ $y_2 \in Im(Y_2)$

find all x_1 and x_2 such

That $x_1 + x_2 = y_1$

$x_1 - x_2 = y_2$

$Im(Y_1, Y_2) = \{$ all y_1 and y_2 That are both even or both odd and

$y_1 + y_2 > 0$ and $y_1 - y_2 > 0 \}$

if This is True

$$x_1 = (y_1 + y_2)/2$$

$$x_2 = (y_1 - y_2)/2$$

All $(y_1, y_2) \in Im(Y_1, Y_2)$ have

$$P(y_1, y_2) = \frac{1}{36}$$

For every $(y_1, y_2) \in \mathrm{Im}(Y_1, Y_2)$

$y_1 = 4 \qquad y_2 = 2$

$\Downarrow$

$x_1 = 3 \qquad x_2 = 1$

$$P_{Y_1 Y_2}(4, 2) = P_{X_1 X_2}(3, 1) = \frac{1}{36}$$

Marginals

$$P_{Y_1}(y_1) = \mathbb{P}(Y_1 = y_1) =$$

$$= \sum_{y_2} \mathbb{P}(Y_1 = y_1 \ \& \ Y_2 = y_2) =$$

$$= \sum_{y_2} P_{Y_1 Y_2}(y_1, y_2)$$

Two Dice:

$$P_{Y_1}(2) = \frac{1}{36}$$

$$P_{Y_1}(3) = \frac{2}{36}$$

$$P_{Y_1}(4) = \frac{3}{36}$$

$$\vdots$$

$$P_{Y_1}(7) = \frac{1}{6}$$

$$P_{Y_1}(8) = \frac{5}{36}$$

Marginal on Y_2

$$P_{Y_2}(y_2) = \sum_{y_1} \mathbb{P}(Y_1 = y_1 \ \& \ Y_2 = y_2)$$

$$P_{Y_2}(-5) = P_{Y_2}(5) = \frac{1}{36}$$

$$\vdots$$

$$P_{Y_2}(0) = \frac{1}{6}$$

A random student

X_1 = passed Calc I

X_2 = passed Lin. Alg.

		X_1: 0	1	P_{X_2}
X_2	0	0.1	0.25	0.35
	1	0.15	0.5	0.65
P_{X_1}		0.25	0.75	1

$P_{X_1X_2}(1,0)$

$$\sum_{x_1,x_2} P_{X_1X_2}(x_1,x_2) = 1$$

Expectations.

$$\mathbb{E}(X_1) = \sum_{x_1} x_1 \, P_{X_1}(x_1) =$$

$$= \sum_{x_1,x_2} x_1 \, P_{X_1X_2}(x_1,x_2)$$

$$\mathbb{E}(X_2) = \sum_{x_1,x_2} x_2 \, P_{X_1X_2}(x_1,x_2)$$

$$\mathbb{E}(h(X_1, X_2)) = \sum_{x_1, x_2} h(x_1, x_2)\, p_{X_1 X_2}(x_1, x_2)$$

$$\mathbb{E}(X_1) = \mathbb{E}(h(X_1, X_2))$$

$$h(x_1, x_2) = x_1$$

$$\mathbb{E}(X_1 X_2) = \sum_{x_1, x_2} x_1 x_2\, p_{X_1 X_2}(x_1, x_2)$$

Assume that for every x_1 and x_2

$$\mathbb{P}(X_1 = x_1 \;\&\; X_2 = x_2) = \mathbb{P}(X_1 = x_1)\,\mathbb{P}(X_2 = x_2)$$

($X_1 \perp\!\!\!\perp X_2$ or X_1 indep. from X_2

or X_1 and X_2 are indep.)

If $X_1 \perp\!\!\!\perp X_2$ I get

$$\mathbb{E}(X_1 X_2) = \sum_{x_1, x_2} x_1 x_2 \, p_{X_1 X_2}(x_1, x_2) =$$

$$= \sum_{x_1} x_1 \, p_{X_1}(x_1) \sum_{x_2} x_2 \, p_{X_2}(x_2) =$$

$$= \mathbb{E}(X_1)\,\mathbb{E}(X_2)$$

$$\operatorname{cov}(X_1, X_2) = \mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\,\mathbb{E}(X_2)$$

Th: if $X_1 \perp\!\!\!\perp X_2 \implies$

$$\operatorname{cov}(X_1, X_2) = 0$$

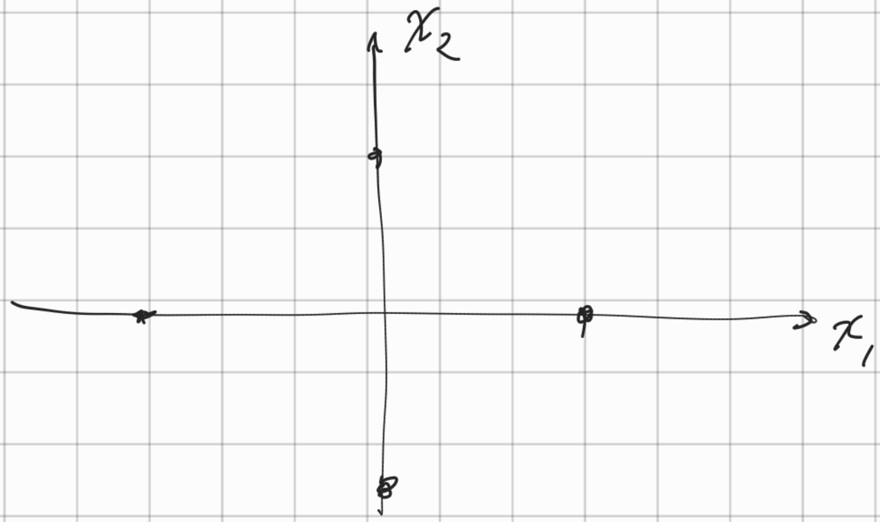
If $\operatorname{cov}(X_1, X_2) = 0$ Then X_1 and X_2 uncorrelated.

X_1 X_2 uncorrelated $\not\Longrightarrow$ X_1 X_2 independent.

$X_1 \quad X_2$

$p(0,1) = p(0,-1) =$

$p(-1,0) = p(1,0) = \frac{1}{4}$



$p_{X_1}(-1) = p_{X_1}(1) = \frac{1}{4} \qquad p_{X_1}(0) = \frac{1}{2}$

$\mathbb{E}(X_1) = \mathbb{E}(X_2) = 0$

$\mathbb{E}(X_1 X_2) = 0$

$\operatorname{cov}(X_1 X_2) = 0$.

$p(0,0) = 0 \quad , \quad p_{X_1}(0)\, p_{X_2}(0) = \frac{1}{4}$

$$\rho_{X_1 X_2} = \frac{\mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)}{\sigma_{X_1} \sigma_{X_2}}$$

$Y_1 = a X_1 + b \qquad a > 0$

$\rho_{Y_1, X_2} = \rho_{X_1 X_2}$

ρ correlation coefficient.

$$-1 \leq \rho \leq 1$$

(Cauchy Schwarz ineq.)

$\rho_{X_1 X_2} = 1 \implies X_2 = a X_1 + b$

$a > 0$

$\rho_{X_1 X_2} = -1 \implies X_2 = a X_1 + b$

$a < 0$

$(0,1) \quad (1,0)$

$(0,0) \quad (0,1) \qquad 1/4$

X Y are r.v.

$X \perp\!\!\!\perp Y$ if and only if

$\forall g, h : \mathbb{R} \to \mathbb{R}$

(1) $\mathbb{E}(g(X)h(Y)) = \mathbb{E}(g(X))\mathbb{E}(h(Y))$

$X \perp\!\!\!\perp Y \implies (1)$

$(1) \implies X \perp\!\!\!\perp Y$

For $a \in \mathrm{Im}(X)$ $b \in \mathrm{Im}(Y)$

$$g(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases} \qquad h(y) = \begin{cases} 1 & y = b \\ 0 & \text{other.} \end{cases}$$

$$\mathbb{E}(g(X)) = \mathbb{P}(X = a)$$

$$\mathbb{E}(h(Y)) = \mathbb{P}(Y = b)$$

$$\mathbb{E}(g(X)h(Y)) = \mathbb{P}(X = a \ \& \ Y = b)$$

$$\mathbb{P}(X = a \ \& \ Y = b) = \mathbb{P}(X = a)\,\mathbb{P}(Y = b)$$

often it is enough to check

(1) for $g(x) = x^n$ $h(y) = y^m$ $\forall n, m$